On the stability of plane shock waves

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SUMMARY

The decay of small perturbations on a plane shock wave propagating along a two-dimensional channel into a fluid at rest is investigated mathematically. The perturbations arise from small departures of the walls from uniform parallel shape or, physically, by placing small obstacles on the otherwise plane parallel walls. An expression for the pressure on a shock wave entering a uniformly, but slowly, diverging channel already exists (given by Chester 1953) as a deduction from the Lighthill (1949) linearized small disturbance theory of flow behind nearly plane shock waves. Using this result, an expression for the pressure distribution produced by the obstacles upon the shock wave is built up as an integral of Fourier type. From this, the shock shape, ξ, is deduced and the decay of the perturbations obtained from an expansion (valid after the disturbances have been reflected many times between the walls) for ξ in descending power of the distance, ζ, travelled by the shock wave. It is shown that the stability properties of the shock wave are qualitatively similar to those discussed in a previous paper (Freeman 1955); the perturbations dying out in an oscillatory manner like $\zeta^{-3/2}$. As before, a Mach number of maximum stability (1·15) exists, the disturbances to the shock wave decaying most rapidly at this Mach number. A modified, but more complicated, expansion for the perturbations, for use when the shock wave Mach number is large, is given in §4.

In particular, the results are derived for the case of symmetrical 'roof top' obstacles. These predictions are compared with data obtained from experiments with similar obstacles on the walls of a shock tube.

1. INTRODUCTION

The ability of plane shock waves, propagating into a stationary fluid, to retain their shape when subjected to small disturbances was considered theoretically in a previous paper (Freeman 1955). This property has been called 'stability'. It was realized however that the model used to demonstrate

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this, namely, a perturbation to the shock wave produced by moving a plane piston impulsively from rest at constant speed into the stationary fluid, would be difficult to reproduce in the laboratory. In order to overcome this disadvantage another model is here studied. In this model, a plane shock wave propagating down a uniform channel into a fluid at rest is disturbed by small protuberances on the channel wall. This model bears a close resemblance to actual experimental conditions when the shock is produced, as is usually the case, in a shock tube, and, in fact, such experiments have been performed by Mr K. C. Lapworth in the Manchester University shock tube. It will be shown that the shock wave qualitatively exhibits similar properties to those described in the preceding paper (Freeman 1955).

Chester (1953) has already considered the problem of a shock propagating along a two-dimensional channel which begins to diverge slowly, and has extended this result to obtain the change in strength of a shock propagating along a channel which changes from one uniform cross-section to another. The solution given below will be a further development of this work obtained, essentially, by considering higher order terms in the expansion given by Chester. In its turn, the Chester solution is a development of the problem, investigated by Lighthill (1949), of a shock wave propagating along a plane wall which suddenly changes in direction by a small angle δ. The fluid is assumed inviscid except in a thin region comprising the shock wave across which the variables of the flow change discontinuously in a manner determined by the Rankine–Hugoniot relations. The disturbances produced at the corner are assumed to be small enough for the equations of the flow behind the shock wave to be linearized. The shock wave is disturbed from its plane form within a region cut off by a cylindrical wavefront, which originating at the corner, expands with the speed of sound behind the undisturbed shock wave and is translated bodily with the velocity of the fluid behind the undisturbed shock wave (see figure 1). When the corner is concave to the flow, this becomes the familiar Mach reflection. Under the conditions stated above, the pressure satisfies the wave equation with constant sound speed and the solution can be obtained in terms of the 'conefield' variables by the use of the Busemann transformation and conformal mapping techniques. Chester has shown that the presence of another wall can be accounted for by considering a system of images in the wall. The solution is rather complicated, being a sum of Lighthill solutions. After the disturbances produced have undergone a sufficiently large number of reflections at the walls, however, an asymptotic expansion for the pressure on the shock wave can be obtained in descending powers of the time. The first term of this expansion is used by Chester in his solution for a change in cross-section of the channel. Further higher order terms, which decay with time, are neglected. It is these terms that we shall consider here. In the case when the channel is uniform along its length, except over a finite range, after which it returns to its original cross-section, the decaying terms alone remain.

We, therefore, consider the propagation of a plane shock wave along a uniform two-dimensional channel with walls a fixed distance apart except over a finite length where the channel width varies slowly in some prescribed manner. Following Chester, we will concern ourselves with the behaviour at large time, that is, after many reflections have taken place at the channel walls. It will also be assumed that sufficient time has elapsed for the shock to have travelled a distance large compared with the length of the non-uniform section of the channel.

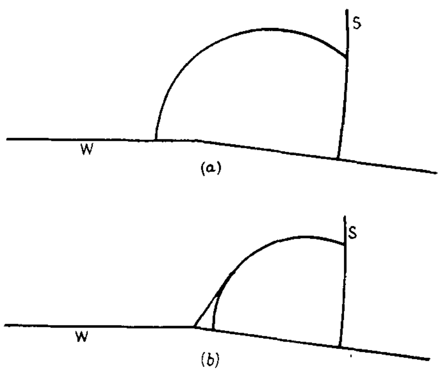

Figure 1. The configuration in the Lighthill problem. S, shock; W, wall. (a) Subsonic; (b) Supersonic.

The perturbations to the flow arising from each element of the disturbing wall are propagated, as pointed out in the previous paper (Freeman 1955), at the speed of sound behind the undisturbed shock as cylindrical waves convected with the fluid. The velocity of the disturbances along the shock wave is the velocity component of these wavefronts along the shock. To the linearized approximation, the velocity of the wavefront of the cylindrical wave produced by a particular element of wall is $a_1\sqrt{\{1-([U-u]/a_1)^2\}}$ ($= a_1 \sin\psi$, say) along the shock, where a_1 is the velocity of sound behind the undisturbed shock and $U-u$ is the velocity of the undisturbed shock relative to the fluid behind it. As these waves are cylindrical in character, we expect a singularity like $\sqrt{\{1-(r/a_1 t)^2\}}$ where r is the distance from the centre of the disturbance produced by the particular element under consideration, and t is the time measured from the instant at which the shock strikes it. But, as has been shown previously (Freeman 1955), the strength of this wavefront is forced to vanish at the shock by the form of the boundary condition there, and the singularity is of higher order. The effect of the non-uniformity of the channel wall is then the sum of all these cylindrical waves and is obtained as a Fourier transform of the individual solutions. At large time, the behaviour of this integral is dominated by the singularity at the cylindrical wavefront. Thus, the shock is perturbed on passing the non-uniformity by disturbances which travel along it with a speed $a_1 \sin\psi$. Due to the singularity at the wavefront, the disturbances decay in an oscillatory manner like $\zeta^{-3/2}$ where ζ is the distance travelled past the obstacle. This rate of decay varies with shock Mach number, decreasing rapidly both for strong and very weak shocks. Between these extremes there lies a Mach number of maximum stability where the perturbations to the shock wave decay most rapidly with the distance travelled by the shock. This has been calculated as $M = 1{\cdot}15$ approximately, which is very close to the value 1·14 given for the first model (Freeman 1955).

An interesting phenomenon occurs when we consider the particular case of symmetrical 'roof top' obstacles placed on the upper and lower walls of the channel (see figure 2). Then, together with the above variation in Mach number, there is a variation with the length of the 'roof top' due to cancellation or reinforcement in varying degrees of the disturbances produced at the corners and vertices. A similar effect occurs in linearized aerofoil theory of supersonic flow for the so-called 'Busemann Biplane'. When the shock wave strikes the 'roof tops', disturbances propagate outwards along it and are reflected at the opposite walls. At the ends of the 'roof top', the corners being concave to the flow, the disturbances will

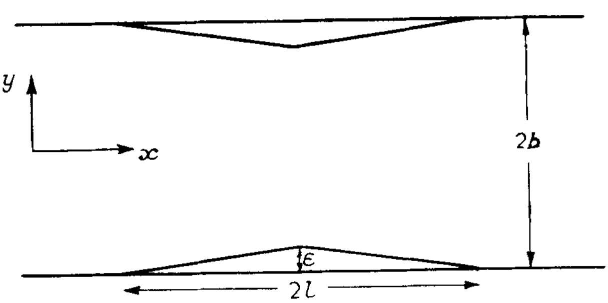

Figure 2. The 'roof top' obstacles in position in the channel.

be compressive ones, whereas at the vertex, the corner being convex to the flow, the disturbance will be expansive. The latter disturbance, according to linear theory, is opposite in sign and twice the magnitude of either of the former. On this account, should the length of the 'roof top' be such that the time taken for the shock to travel between corner and vertex is an integral number of times that for the disturbances on the shock to travel across the channel, cancellation of these disturbances will occur and the decay will be faster than indicated above. In a similar manner, maximum disturbance is produced when the time taken for the shock to move from corner to corner is an odd integral multiple of that for the disturbances to cross the channel. These conditions may be written $Ub/a_1 l \sin\psi = 2n$ and $Ub/a_1 l \sin\psi = 2n+1$, where n is an integer, U is the velocity of the shock wave, $2l$ is the length of the 'roof top' and $2b$ the width of the channel.

In § 5, the results obtained by Lapworth (1956) from experiments on 'roof top' obstacles situated on the walls of a shock tube are briefly discussed. These results indicate that the theory developed in this paper gives a good qualitative picture of actual conditions. The oscillatory nature of the perturbations and their rate of decay are in good agreement although the actual magnitude of the perturbation is somewhat less than predicted. This discrepancy might be expected however since the present theory is only a linear approximation to the true state of affairs.

In conclusion, a comparison of the present model with the one discussed previously (Freeman 1955) makes it clear that the behaviour of the shock wave is to some extent independent of conditions prevailing in the flow field behind. A theory making this assumption has been put forward by Whitham (1957) to take into account non-linear effects.

2. Extension of Chester solution

The theory of Lighthill (1949) has shown that the flow behind a plane shock moving along a plane wall which changes in direction by a small angle δ is disturbed within a circle expanding with the velocity of sound behind the shock and whose centre moves with the velocity of the fluid behind the shock originally being situated at the corner (see figure 2). These velocities are, to this approximation, constant and equal to their value behind the undisturbed shock wave. In the case of supersonic flow behind the shock, there is an additional region of disturbance bounded by the tangent from the corner to this circle (figure 2(*b*)). The shock is disturbed from its plane form in the region cut off by this circle.

The pressure within the circle is given by

$$(p_1 - p_0)\,\delta p(X, Y), \tag{2.1}$$

where

$$\frac{\partial p}{\partial y_1} + i\frac{\partial p}{\partial x_1} = \frac{C[D(z_1 - 1 + \gamma^2) - 1]\sec\psi}{(z_1^2 - 1)^{1/2}(z_1 - 1 + \gamma^2)(\alpha - i(z_1 - 1)^{1/2})(\beta - i(z_1 - 1)^{1/2})} \tag{2.2}$$

with $\cos\psi = [(M^2+5)/(7M^2-1)]^{1/2}$. The constants α, β, γ, C and D are functions of M defined by Lighthill (1949). M is the Mach number of the shock; a_0, p_0, ρ_0 and a_1, p_1, ρ_1 are the sound speed, pressure and density in front and behind the shock respectively. The $z_1(= x_1 + iy_1)$-plane is related to the physical (x, y)-plane with origin at the corner and x-axis in the direction of propagation by

$$z_1 = -\frac{1}{2}\left[\left\{\frac{\rho e^{i(\theta+\psi)} - 1}{\rho e^{i\theta} - e^{i\psi}}\right\}^2 + \left\{\frac{\rho e^{i\theta} - e^{i\psi}}{\rho e^{i(\theta+\psi)} - 1}\right\}^2\right] \tag{2.3}$$

and

$$\frac{x - q_1 t}{a_1 t} + i\frac{y}{a_1 t} = X + iY = \frac{2\rho e^{i\theta}}{1+\rho^2},$$

where q_1 is the velocity of the fluid behind the undisturbed shock wave. The disturbed region in the (x, y)-plane is mapped into the upper half plane. The disturbed shock, which is given by $x = Ut$ or $X = \cos\psi$, becomes

$$x_1 = \frac{1 + (Y/\sin\psi)^2}{1 - (Y/\sin\psi)^2}, \qquad y_1 = 0, \tag{2.4}$$

with

$$0 < Y < \sin\psi.$$

The point $x_1 = 1$, $y_1 = 0$ is the intersection of the shock and wall and $x_1 = \infty$, $y_1 = 0$ the intersection with the circular wavefront. From (2.2),

$$\frac{\partial p}{\partial x_1} = \frac{C[D(x_1-1+\gamma^2)-1](\alpha+\beta)\sec\psi}{(x_1+1)^{1/2}(x_1-1+\gamma^2)(x_1-1+\beta^2)(x_1-1+\alpha^2)} \tag{2.5}$$

on the shock.

Chester (1953) has shown that the problem of a shock moving along a channel of width $2b$ which begins to diverge uniformly at a small angle 2δ can be solved in terms of this solution. The effect of enclosing the shock between two walls is to reflect the waves produced at the corners repeatedly between the two walls. Since this wave motion is already of order δ, however, the effect of the divergence of the walls will be a second order effect in δ in the reflections, and hence these reflections may be considered to take place at plane walls. Suitable images can be placed outside the walls to produce the required reflections. The number of these images that influence the shock will grow with time. The pressure in the case of a channel can therefore be written

$$\delta(p_1-p_0)P(x,y,t),$$

with

$$P = \sum_n \left\{p\left(\frac{x-q_1 t}{a_1 t}, \frac{(2n+1)b-y}{a_1 t}\right)+p\left(\frac{x-q_1 t}{a_1 t}, \frac{(2n+1)b+y}{a_1 t}\right)\right\}, \tag{2.6}$$

where x is measured along the channel and y from the centre of the channel, the summation being taken over all the images that influence the shock. By using Dirichlet's summation formula, Chester writes this, in a form more convenient for asymptotic estimation, as

$$\begin{aligned} P = {} & \tfrac{1}{2}p(X,(b-y)/a_1 t)+\tfrac{1}{2}p(X,(b+y)/a_1 t)+ \\ & +\frac{a_1 t}{2b}\sum_{n=-\infty}^{\infty}\left[\int_{(b-y)/a_1 t}^{(1-X^2)^{1/2}}(-1)^n p(X,Y)e^{n\pi i(a_1 tY+y)/b}\,dY+\right. \\ & \left.+\int_{(b+y)/a_1 t}^{(1-X^2)^{1/2}}(-1)^n p(X,Y)e^{n\pi i(a_1 tY-y)/b}\,dY\right], \end{aligned} \tag{2.7}$$

where $p(X, Y)$ is given by (2.2).

If we now consider pressure variations on the shock alone, then

$$\begin{aligned} P = {} & \tfrac{1}{2}p\left(\cos\psi,\frac{b-y}{a_1 t}\right)+\tfrac{1}{2}p\left(\cos\psi,\frac{b+y}{a_1 t}\right)+ \\ & +\frac{a_1 t}{2b}\sum_{n=-\infty}^{\infty}\left\{\int_{(b-y)/a_1 t}^{\sin\psi}(-1)^n p(\cos\psi,Y)e^{n\pi i(a_1 tY+y)/b}\,dY\right. \\ & \left.+\int_{(b+y)/a_1 t}^{\sin\psi}(-1)^n p(\cos\psi,Y)e^{n\pi i(a_1 tY-y)/b}\,dY\right\}. \end{aligned} \tag{2.8}$$

It will be sufficient to consider the behaviour of this function for large time.

The behaviour of the first two terms will simply be determined by expanding p as a function of Y. The integrals within the sum will however be dominated by the singularities of p in the range of integration. It will be shown as in the previous paper (Freeman 1955) that the singularity at the shock–wavefront intersection is dominant.

Let us now consider the function $p(\cos\psi, Y)$ in more detail. Near the wall, when $Y = 0$, $x_1 = 0$, (2.5) together with (2.4) states that

$$p = \text{const.} + O(Y^2).$$

Thus $\partial p/\partial Y = 0$ at $Y = 0$ and also

$$p(\cos\psi, (b \pm y)/a_1 t) = \text{const.} + O((b \pm y)^2/a_1^2 t^2). \tag{2.9}$$

At the shock–wavefront intersection $Y = \sin\psi$, or $x_1 = \infty$, then, from (2.5),

$$\frac{\partial p}{\partial x_1} \sim \frac{CD(\alpha+\beta)\sec\psi}{x_1^{5/2}}. \tag{2.10}$$

Consider now a general term of the sum in (2.8). For $n \neq 0$ we have

$$\begin{aligned} J_n &= \frac{a_1 t}{2b} e^{\pm in\pi y/b} \int_{(b\mp y)/a_1 t}^{\sin\psi} (-1)^n p(\cos\psi, Y) e^{in\pi a_1 tY/b}\, dY \\ &= \frac{(-1)^n}{2n\pi i}\left\{A - e^{\pm in\pi y/b}\int_{(b\mp y)/a_1 t}^{\sin\psi} \frac{\partial p}{\partial Y} e^{in\pi a_1 tY/b}\, dY\right\} + O\left(\frac{1}{t^2}\right), \end{aligned} \tag{2.11}$$

using (2.9) and (2.5), where A is a constant. Also, since $\partial p/\partial Y = O(Y)$ as $Y \to 0$, it follows that

$$J_n = B_n - \frac{(-1)^n e^{\pm in\pi y/b}}{2n\pi i}\int_0^{\sin\psi} \frac{\partial p}{\partial Y} e^{in\pi a_1 tY/b} + O\left(\frac{1}{t^2}\right), \tag{2.12}$$

where B_n is a constant and the integration is now along the shock from the wall to the wavefront. It follows also that the above integral is dominated by the singularity at the upper limit. The integrand is

$$\begin{aligned} &\frac{\partial p}{\partial x_1}\left(\frac{dx_1}{dY}\right)_{\text{shock}} e^{in\pi a_1 tY/b} \\ &\sim CD(\alpha+\beta)\sec\psi \operatorname{cosec}\psi (1 - Y\operatorname{cosec}\psi)^{1/2} e^{in\pi a_1 tY/b} \end{aligned} \tag{2.13}$$

near $Y = \sin\psi$, and so the integral may be written

$$CD(\alpha+\beta)\sec\psi\operatorname{cosec}\psi \int^{\sin\psi} (1 - Y\operatorname{cosec}\psi)^{1/2} e^{in\pi a_1 tY/b}\, dY + O\left(\frac{1}{t^2}\right).$$

Using the formula for the asymptotic expansion of a Fourier integral (see, for example, Freeman 1955), we obtain

$$\begin{aligned} J_n \sim B_n + \frac{(-1)^n CD(\alpha+\beta)\sec\psi}{4|n|\sqrt{\pi}}\left|\frac{b}{n\pi a_1 t\sin\psi}\right|^{3/2} \times \\ \times \exp\left[\frac{in\pi}{b}\{a_1 t\sin\psi \pm y\} + \frac{i\pi}{4}\theta_n\right], \end{aligned} \tag{2.14}$$

where
$$\begin{aligned} \theta_n &= -1 \quad (n > 0), \\ &= +1 \quad (n < 0), \end{aligned}$$

for t large. Substituting in (2.8), we have

$$\begin{aligned} P = \frac{a_1 t}{b}\int_0^{\sin\psi} p(\cos\psi, Y)\, dY + B + \\ + \frac{CD(\alpha+\beta)\sec\psi}{2\pi^2}\left[\left(\frac{b}{a_1 t\sin\psi}\right)^{3/2} \sum_{n=-\infty}^{\infty}{}' \frac{(-1)^n}{|n|^{5/2}} \times \right. \\ \left. \times \exp\left\{\frac{in\pi a_1 t\sin\psi}{b} + \frac{i\pi}{4}\theta_n\right\}\cos\frac{n\pi y}{b}\right] + O\left(\frac{1}{t^2}\right), \end{aligned} \tag{2.15}$$

where $\sum'$ denotes that there is no term $n = 0$ in the summation, and B is a constant. The integral has been calculated by Chester, who denotes it by $K(U/a_1)$, and hence

$$P = \frac{KUt}{b} + B + \frac{CD(\alpha+\beta)\sec\psi}{\pi^2}\left(\frac{b}{a_1 t \sin\psi}\right)^{3/2} \times$$
$$\times \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{5/2}} \cos\left(\frac{n\pi a_1 t\sin\psi}{b} - \frac{\pi}{4}\right)\cos\frac{n\pi y}{b} + O\left(\frac{1}{t^2}\right). \quad (2.16)$$

Now, let us consider a channel of more general cross-section defined by $y = b+f(x)$, $y = -b-f(x)$, where $f(x)$ is non-zero for some range of x and $f'(x)$ is small. In fact, we will consider the case of $f(x)$ non-zero only in a finite range, that is, the effects of change in shock strength due to changes in width of channel, which have already been studied by Chester, will not be considered. The solution of the problem in which the channel begins to diverge uniformly can now be generalized to this new channel shape by considering a change of slope $df'(\eta)$ at time $t = \eta/U$ and distance $x = \eta$. Thus the value of P for the new problem will be

$$P_1 = \int P\left(t - \frac{\eta}{U}, y\right) df'(\eta), \quad (2.17)$$

the integral being taken over the range in which f is non-zero. For sufficiently large distances away from the obstacle in the channel, the powers of $t-(\eta/U)$ become t and thus,

$$P_1 = \frac{CD(\alpha+\beta)\sec\psi}{\pi^2}\left(\frac{b}{a_1 t\sin\psi}\right)^{3/2}$$
$$\times \sum_{n=1}^{\infty} \int \frac{(-1)^n}{n^{5/2}} \cos\left[\frac{n\pi a_1\sin\psi}{b}\left(t - \frac{\eta}{U}\right) - \frac{\pi}{4}\right) \times$$
$$\times \cos\frac{n\pi y}{b}\, df'(\eta) + O\left(\frac{1}{t^2}\right). \quad (2.18)$$

Now if ξ is the perturbation of the shock wave from its plane form, then the perturbation to the pressure is $\frac{5}{3}Ma_0\rho_0(\partial\xi/\partial t)$, to a linear approximation, from the shock equation ($\gamma = 1{\cdot}4$). Hence the perturbation to the shock shape may be written

$$\frac{\xi}{b} = \frac{3CD(\alpha+\beta)}{5\pi^3 M\cos\psi\sin\psi}\left(\frac{p_1-p_0}{a_1 a_0\rho_0}\right)\left(\frac{b}{a_1 t\sin\psi}\right)^{3/2} \times$$
$$\times \sum_{n=1}^{\infty} \int \frac{(-1)^n}{n^{7/2}} \sin\left[\frac{n\pi a_1\sin\psi}{b}\left(t - \frac{\eta}{U}\right) - \frac{\pi}{4}\right]\cos\frac{n\pi y}{b}\, df'(\eta) + O\left(\frac{1}{t^2}\right). \quad (2.19)$$

This result indicates, as has been shown previously (Freeman 1955), that the perturbations to the shock wave decay at a rate proportional to the inverse three-halves power of the time, or, alternatively, of the distance travelled by the shock wave past the obstacle. The variation of the coefficient in (2.19) with Mach number will be discussed in more detail in § 3.

In passing, it might be observed that the modification to equation (2.19) for the case of a channel which increases in width by a small amount $2c$ is obtained by adding a term

$$-\frac{3(p_1-p_0)K_c}{5\rho_0 U^2 b}\cdot\frac{Ut}{b}$$

to the right-hand side.

3. THE 'ROOF TOP' DISTURBANCE

In order to obtain more detailed knowledge of this phenomenon, we will now study a particular form of the function $f(x)$—the so-called 'roof top' (see figure 2). Let

$$\left.\begin{aligned} f(\eta) &= -\left(1+\frac{\eta}{l}\right)\epsilon \qquad (-l<\eta<0), \\ &= -\left(1-\frac{\eta}{l}\right)\epsilon \qquad (0<\eta<l). \end{aligned}\right\} \tag{3.1}$$

Then, using (2.17), we obtain

$$P_1 = \frac{\epsilon}{l}\left\{2P(t,y)-P\left(t-\frac{l}{U},\ y\right)-P\left(t+\frac{l}{U},\ y\right)\right\}. \tag{3.2}$$

The total disturbance is made up of a sum, therefore, of the individual disturbances at the corners and vertex of the 'roof top', due regard being taken of their origin in time. Hence, using (2.19) and (3.2),

$$\begin{aligned} \frac{\xi}{b} = \frac{\epsilon}{l}\Omega_1\left(\frac{b}{a_1 t\sin\psi}\right)^{3/2} &\times \\ \times\Big[2E\Big(\frac{\pi}{b}(a_1 t\sin\psi+y)\Big)&+2E\Big(\frac{\pi}{b}(a_1 t\sin\psi-y)\Big)- \\ -E\Big(\frac{\pi}{b}(a_1\sin\psi\{t+(l/U)\}+y)\Big)&-E\Big(\frac{\pi}{b}(a_1\sin\psi\{t+(l/U)\}-y)\Big)- \\ -E\Big(\frac{\pi}{b}(a_1\sin\psi\{t-(l/U)\}+y)\Big)&-E\Big(\frac{\pi}{b}(a_1\sin\psi\{t-(l/U)\}-y)\Big)\Big], \end{aligned} \tag{3.3}$$

where $$E(z) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^{7/2}}(\sin nz-\cos nz)$$

and $$\Omega_1 = \frac{3CD(\alpha+\beta)(p_1-p_0)}{10\pi^3\sqrt{2M}\cos\psi\sin\psi a_1 a_0\rho_0}.$$

If $\zeta = Ut$, the distance travelled by the shock, then

$$\frac{\xi}{b} = \frac{\epsilon}{l}\frac{\Omega}{(\zeta/b)^{3/2}}\left[F\left(\frac{\pi}{b}\left\{\frac{a_1\sin\psi}{U}\zeta+y\right\}\right)+F\left(\frac{\pi}{b}\left\{\frac{a_1\sin\psi}{U}\zeta-y\right\}\right)\right],$$

where $$F(z) = 2E(z)-E\left(z+\frac{\pi a_1 l\sin\psi}{Ub}\right)-E\left(z-\frac{\pi a_1 l\sin\psi}{Ub}\right) \tag{3.4}$$

and $$\Omega = \left(\frac{U}{a_1\sin\psi}\right)^{3/2}\Omega_1.$$

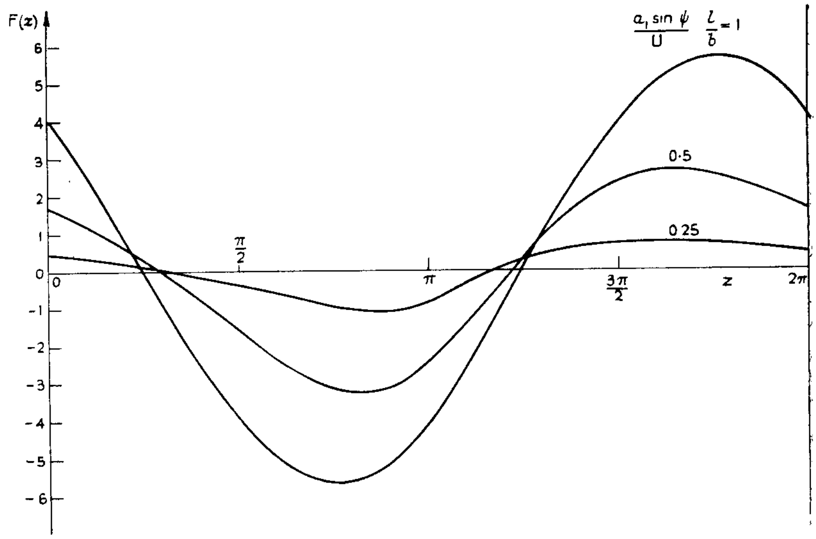

Figure 3. The function $F(z)$.

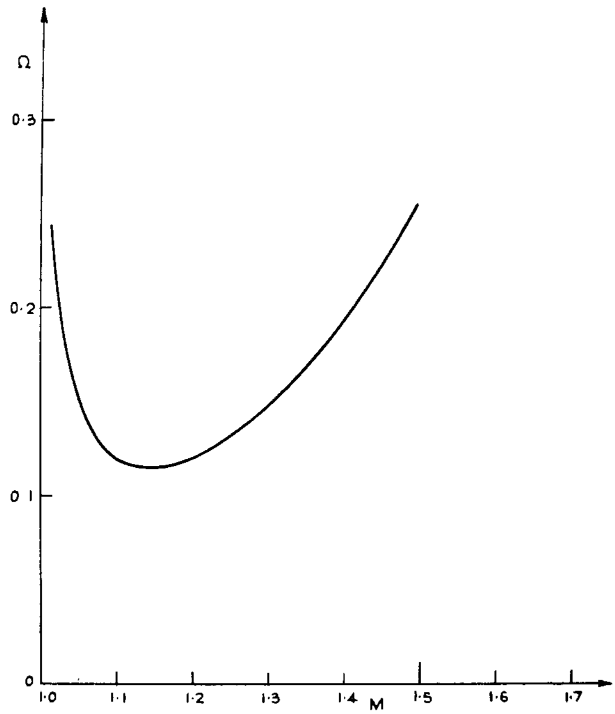

Figure 4. The coefficient Ω as a function of M.

The function $E(z)$ was computed by making use of the relation

$$\sum_{n=1}^{\infty} \frac{e^{inx}}{n^{s+1}} = \frac{1}{s!}\int_0^{\infty} \frac{z^s\,dx}{e^{z-ix}-1}. \tag{3.5}$$

The function $F(z)$ is plotted in figure 3 for various values of $la_1 \sin\psi/Ub$. $F(z)$ is periodic with period 2π. For $la_1 \sin\psi/Ub = 2n$, where n is an integer, the function is identically zero. This condition may be written $l/U = (2b/a_1 \sin\psi)n$, which states that the time taken for the shock to travel between a corner and the vertex of the 'roof top' is an integral multiple of the time taken for a disturbance to travel along the shock across the channel. If this is so, the initial term of our expansion will vanish and higher order terms will dominate, the shock perturbations decaying more rapidly. The function F is a maximum for values of $(a_1 l \sin\psi/Ub)$ at which $(l/U) = (b/a_1 \sin\psi)(2n+1)$. These values correspond to configurations in which the disturbances produced at the vertex and corners reinforce each other. The coefficient Ω is shown in figure 4. The similarity between this figure and figure 5 of the earlier paper (Freeman 1955) is immediately obvious. The minimum at $M = 1{\cdot}15$ is very close to the one shown there. The perturbations to shock waves of this Mach number decay most rapidly with the distance travelled by the shock wave.

4. The variation at high Mach number

The behaviour of the various constants for large Mach number M may be summarized as follows:

$$\cos\psi \sim \surd(\tfrac{1}{7}), \qquad \sin\psi \sim \surd(\tfrac{6}{7}), \qquad \alpha \sim 2\surd(\tfrac{2}{7})M^2, \qquad \beta \sim \surd(\tfrac{7}{2}),$$
$$\gamma \sim \surd(\tfrac{1}{2}), \qquad C \sim 35\surd(\tfrac{6}{7})M^2/\pi, \qquad D \sim \tfrac{1}{8}. \tag{4.1}$$

The correct approximation to equation (2.5) for large M and large x_1 is, therefore,

$$\frac{\partial p}{\partial x_1} \sim \left(\frac{CD}{\alpha}\right) \frac{[1+(\beta/\alpha)]\sec\psi}{x_1^{3/2}\{1+[(x_1-1)/\alpha^2]\}}, \tag{4.2}$$

since α is no longer small. It will be observed that as $\alpha \to \infty$ then $(\partial p/\partial x_1) \sim x^{-3/2}$ and the form of the singularity that dominates the integral (2.13) has changed. When α is large but finite, we must consider a form of expansion in which the pole $x_1 = -\alpha^2$ lies close to the branch point. The method used is due to Clemmow (1950). From (4.2),

$$\frac{\partial p}{\partial x_1} \sim \frac{\Theta z^{5/2}}{z+\alpha^{-2}} \tag{4.3}$$

near $z = 0$, where $z = 1-(Y/\sin\psi)$ and $\Theta = CD(\alpha+\beta)\sec\psi/\alpha^2$. Whence, using (2.4),

$$\frac{\partial p}{\partial Y} \sim \frac{\Theta z^{1/2}}{\sin\psi(z+\alpha^{-2})}, \tag{4.4}$$

and thus for large M,

$$\frac{\partial p}{\partial Y} \sim \frac{\Theta}{\sin\psi\{1-(Y/\sin\psi)\}^{1/2}}$$

as opposed to
$$\frac{\partial p}{\partial Y} \sim \frac{\alpha^2 \Theta}{\sin\psi\{1-(Y/\sin\psi)\}^{1/2}}$$
for smaller M. This in turn would give a slower decay, like $t^{-1/2}$, to the shock perturbations for large M. For large M, therefore, we require an expression for
$$\int_0^\infty \frac{z^{1/2}}{z+\alpha^{-2}} e^{in\pi a_1 tz \sin\psi/b}\, dz \tag{4.5}$$
in place of the expansion of (2.13) since, as before, the main contribution will come from near $z = 0$. Thus, as in § 2, the pressure due to a slightly diverging channel is
$$P = -\frac{KUt}{b} + B + $$
$$+\Theta \sum_{n=-\infty}^{\infty}{}' \frac{(-1)^n}{n\pi i}\left[\int_0^\infty \frac{z^{1/2}}{z+\alpha^{-2}} e^{in\pi a_1 t \sin\psi z/b}\, dz\right] e^{-in\pi a_1 t \sin\psi/b} + O\left(\frac{1}{t^2}\right). \tag{4.6}$$
The integral enclosed in the square brackets may be written
$$\left(\frac{b}{na_1 t\sin\psi}\right)^{1/2} I^*\left(\frac{n\pi a_1 t \sin\psi}{\alpha^2 b}\right) e^{\frac{1}{4}i\pi} \quad \text{for } n > 0$$
and
$$\left(\frac{b}{|n| a_1 t\sin\psi}\right)^{1/2} I\left(\frac{|n|\pi a_1 t \sin\psi}{\alpha^2 b}\right) e^{-\frac{1}{4}i\pi} \quad \text{for } n < 0,$$
where * denotes the complex conjugate and
$$I(z) = 1 - \sqrt{(\pi z)} e^{iz}\left\{e^{\frac{1}{4}i\pi} - \frac{2i}{\sqrt{\pi}} \int_0^{\sqrt{z}} e^{-iv^2}\, dv\right\}. \tag{4.7}$$
As in (2.19), the shock shape is then given, for a channel varying in width, by
$$\frac{\xi}{b} = \frac{\Lambda}{(\zeta/b)^{1/2}} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{5/2}} \cos\frac{n\pi y}{b} \times$$
$$\times \int\left[I^*\left(\frac{n\pi a_1 \sin\psi}{\alpha^2 b}\left\{t - \frac{\eta}{U}\right\}\right) e^{-in\pi a_1 (t-\eta/U)/b + \frac{1}{4}i\pi} - \right.$$
$$\left. - I\left(\frac{n\pi a_1 \sin\psi}{\alpha^2 b}\left\{t - \frac{\eta}{U}\right\}\right) e^{in\pi a_1 (t-\eta/U)/b - \frac{1}{4}i\pi}\right] df'(\eta), \tag{4.8}$$
where Λ is given in table 1. And, in a similar manner to § 3, the solution for a 'roof top' obstacle can be obtained.

M	Λ
1·5	0·102
2·0	0·128
3·0	0·168
∞	0·234

Table 1.

5. Experimental results

The model discussed in the previous section bears a close resemblance to actual experimental conditions in a rectangular shock tube when 'roof top' protuberances are placed on two opposite walls. Such experiments have been undertaken by Mr K. C. Lapworth (1956) in the shock tube at the Mechanics of Fluids Department, Manchester University. The original intention was to make a comprehensive study of the results obtained from the preceding theory. However, the results obtained so far fall somewhat short of this ideal due to the inability to reproduce the theoretical model as closely as would be desired.

In theory, the perturbations to the shock wave are assumed infinitesimally small—a consequence of the linearization of the problem. In practice, it is necessary to produce disturbances which are large enough to be measured. Since the perturbations are decaying all the time, this will always limit the distance downstream at which one can observe the perturbations to the shock wave. As the theoretical result is given by an asymptotic expansion in inverse powers of the distance, this can seriously limit any comparison between theory and experiment by not allowing time for the regime in which this expansion is valid to be set up. In fact, however, the main features of the asymptotic form of decay would seem to be apparent earlier than one might expect theoretically, at least for Mach numbers not too small.

Also, in the theoretical considerations viscous effects are neglected entirely except in so far as they occur in the idealized form of the shock wave. In practice, viscous effects are important especially around the corners of the 'roof tops', producing separation and the formation of a wake from the vertex. As any viscous effects produced at the wedges will take time to develop after the shock has passed, it will take time for such disturbances to grow large enough to influence the shock. It is thought that it is this which puts an even greater restriction on the distance downstream at which measurements of the perturbations under theoretical flow conditions can be made. For, after a time, the perturbations start to increase due probably to the wake formation influencing the shock wave.

The experiments were made at three shock Mach numbers, 1·60, 1·41 and 1·16. The shock wave was photographed using Schlieren techniques at various distances after passing the 'roof top' system. As it was difficult to decide where the undisturbed position of the shock wave would be and hence to measure ξ directly, a measure of the perturbation which was independent of this was used. This will be called 'the total perturbation' and denoted by ξ_T. ξ_T is the sum, without regard to sign, of the maximum and minimum values of ξ along the shock wave at any particular time. The Mach numbers 1·41 and 1·60 give disturbances to the shock wave which are near the maximum for the 'roof top' system considered, and hence, for these, it is sufficiently accurate to approximate the function $F(z)$ by a sinusoidal function of the appropriate amplitude and period. The total perturbation ξ_T can then be written in the form

$$\frac{\xi}{2b} = \frac{G}{(\zeta/2b)^n} |\sin(mx+p)|, \tag{5.1}$$

from equation (3.4), where the values of n, G, m and p are functions of Mach number and are given in table 2.

For the Mach number 1·16, it was found difficult to compare the values with theory in any way. This may be due to the fact that the asymptotic behaviour will take a longer time to establish itself for lower Mach numbers and those results will therefore not be discussed further. For the two Mach numbers 1·41 and 1·60, Lapworth (1956) has attempted to fit the experimental results for a formula of the type (5.1). Estimates of m and p are obtained from the zeros of the function ξ_T and then G and n are obtained by applying a least squares fit. In order to decide at what point the perturbations began to be influenced by the disturbances in the flow produced by separation at the roof tops, the data were analysed using varying numbers of results dispensing one at a time with the later results until a fairly consistent set of values on n and G were obtained. An average of these values is given in table 3.

M	n	m	p	G
1·41	1·5	3·39	−3·91	0·157
1·60	1·5	3·44	−3·91	0·273

Table 2.

M	n	m	p	G
1·41	1·45	3·31	−2·3	0·066
1·60	1·65	3·31	−1·9	0·090

Table 3.

A comparison of tables 2 and 3 shows that even with the difficulties encountered, the decay of the perturbations is in fair qualitative agreement with theoretical predictions. The actual perturbations are two to three times smaller than those predicted but this might be expected from a linear theory. Non-linear effects would interfere with the reinforcement or cancellation of the sets of disturbances from the two roof tops and also cause the disturbances to interact among themselves. The compressive disturbances would tend to concentrate themselves into shock waves whereas the expansive ones would spread out and interact with these waves. A non-linear theory, such as that developed by Whitham (1957), would be required to describe these effects. The resulting perturbations would thus be reduced by a further cancelling of the individual disturbances. The poor agreement of the phase difference p may again be due to the non-asymptotic nature of the flow.

In view of the reason for the breakdown of the theoretical flow conditions in practice, it would seem difficult to modify this model in any simple way to overcome this defect. To make any more detailed study of the phenomena, it would seem that a different shaped obstacle not so prone to separation difficulties would have to be used.

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